

Character Correspondences in Solvable Groups*

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1. INTRODUCTION

Our concern in this paper is with the ordinary (complex number) character theory of solvable groups. If the lengths of some of the earlier papers on this subject [1–3, 8] are a reliable indication, this is a surprisingly complicated and nontrivial theory. Nevertheless, some of the important results are considerably less deep than one would be led to believe by the number of pages involved.

This paper contains no new theorems and few new ideas. It is addressed to the reader who, while familiar with character theory, has not mastered the literature on characters of solvable groups. Its purpose is to give such a reader a feeling for the subject and to provide him with accessible proofs for a few of the main results.

Although we certainly do not attempt to prove the strongest or most general results possible (or known), nevertheless, the paper does contain a few items which have application outside of solvable groups. In particular, we have included (in Section 4) a brief exposition of tensor induction. Although our main result could have been proved using an *ad hoc* argument in place of this general technique, the author is now convinced (after considerable initial resistance) that tensor induction is valuable and deserves further exposure.

Naturally, a key idea in the study of characters of solvable groups is to consider the relationships between characters and chief sections. In particular, suppose that K/L is an abelian chief section of G and let $\theta \in \text{Irr}(K)$ be invariant in G . Then according to the “going down” theorem (Theorem 6.18 of [9]), one of the following must occur:

- (a) θ_L is irreducible.
- (b) $\theta_L = \sum_{i=1}^t \varphi_i$, where the $\varphi_i \in \text{Irr}(L)$ are distinct and $t = |K : L|$.
- (c) $\theta_L = e\varphi$, where $\varphi \in \text{Irr}(L)$ and $e^2 = |K : L|$.

Depending, of course, on the specific problem under consideration, if

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either (a) or (b) occurs, it is generally relatively easy to proceed. (For instance, if (b) holds, then the inertia group $T = I_G(\varphi_1)$ is proper, and an inductive hypothesis might be applied.)

It is case (c) which is most difficult (and interesting) and to which most of our attention will be directed. In this case, we have the following:

THEOREM A. *Let K/L be a chief factor of the solvable group G . Suppose $\theta \in \text{Irr}(K)$ is invariant in G and assume that $\theta_L = e\varphi$, where $\varphi \in \text{Irr}(L)$ and $e^2 = |K : L|$. Then there exists $H \subseteq G$ and a bijection $\pi: \text{Irr}(H | \varphi) \rightarrow \text{Irr}(G | \theta)$ such that*

- (a) $KH = G, K \cap H = L,$
- (b) $\xi^\pi(1) = e\xi(1)$ for all $\xi \in \text{Irr}(H | \varphi).$

The notation $\text{Irr}(G | \theta)$ here means $\{\chi \in \text{Irr}(G) \mid [\chi_K, \theta] \neq 0\}$ and similarly, $\text{Irr}(H | \varphi)$ is the set of irreducible characters of H which lie over φ .

Actually, quite a bit more can be said in the situation of Theorem A. For instance, if $|K/L|$ is odd, one can list further properties (c), (d),... such that H will be unique up to conjugacy and π will be unique. Furthermore, if $|K/L|$ is odd, one need not assume that it is chief; it suffices for it to be abelian. Also, G need not be solvable in this case. The proofs of these assertions constitute the bulk of [2, 8].

If K/L is a 2-group in Theorem A, no uniqueness result for the map π is possible. The first published proof of a result which includes this case is in E. C. Dade's paper [3] although he had obtained results of this nature much earlier in his mimeographed notes [1].

Theorem A is not quite strong enough for the applications we have in mind and so we wish to weaken the hypothesis that K/L is chief.

DEFINITION. Let $L, K \triangleleft G$ with $L \subseteq K$ and K/L abelian. We say that K/L is a *strong* section of G if there exists $N \triangleleft G$ such that the group $S \cong N/C_N(K/L)$ of automorphisms of K/L induced by N satisfies

- (i) $C_{K/L}(S) = 1,$
- (ii) $(|S|, |K/L|) = 1,$
- (iii) S is solvable.

Note that K/L is strong in G iff it is strong in G/L . Also, if G is solvable and K/L is a noncentral chief factor, then K/L is necessarily strong. To see this, let $C = C_G(K/L) < G$ and choose N so that N/C is a chief factor of G . It is then routine to check that $S = N/C$ satisfies the conditions. As a consequence, we see that the next result, which is the main theorem of this paper, includes Theorem A as a special case.

The following is essentially Theorem 5.10 of [3].

THEOREM B. *Assume that K/L is a strong section of G . Let $\theta \in \text{Irr}(K)$ be invariant in G and suppose $\theta_L = e\varphi$, where $e^2 = |K:L|$ and $\varphi \in \text{Irr}(L)$. Then there exists $H \subseteq G$ and a bijection $\pi: \text{Irr}(H|\varphi) \rightarrow \text{Irr}(G|\theta)$ such that*

- (a) $KH = G$ and $K \cap H = L$,
- (b) $\xi^\pi(1) = e\xi(1)$ for $\xi \in \text{Irr}(H|\varphi)$.

As was indicated earlier, the case where $|K/L|$ is even poses special difficulties. One purpose of this paper, beyond that of the exposition of known results, is to present a significant simplification of Dade's proof for this case. Since K/L is strong, we have a certain group $S \cong N/C_N(K/L)$ which we consider. Our simplification exploits the fact that S is solvable. In fact, the theorem would remain true if the assumption that S is solvable were dropped from the definition of a strong section. This is so because if $|K/L|$ is odd, other techniques are available to prove the result (for instance [8]) and if $|K/L|$ is even, then $|S|$ is odd by condition (ii) of the definition, and thus S is solvable by the Feit-Thompson theorem. Here, we are *assuming* that S is solvable, and so we need not consider separately the cases $|K/L|$ odd and $|K/L|$ even. (It should be pointed out that in [3], Dade neither assumes that S is solvable nor uses the odd order theorem.)

The observation that the solvability of S could be used to simplify the argument in [3] was made by the author and independently by R. B. Howlett. (Howlett's version [7] includes what is essentially the hard part of the proof of Theorem B.)

This paper grew out of a series of lectures I gave at the University of Warwick, England, in the spring of 1977, and I will take this opportunity to thank the Mathematics Institute there for their hospitality. Thanks are also due to R. B. Howlett for giving me a preliminary version of his paper (which has influenced the final form of this work) and to E. C. Dade and T. R. Berger whose letters and conversations (respectively) were invaluable.

2. APPLICATIONS

Although it may be somewhat unorthodox to present applications of a theory before discussing the theory itself, it seems appropriate to do so in this case. Theorem B and its proof are rather technical, and the reader may need to be convinced that they are of any value. It is hoped that the two theorems below will provide the necessary motivation.

THEOREM 2.1. *Let A act on G and assume that G is solvable and that $(|A|, |G|) = 1$. Then the number of A -invariant irreducible characters of G is equal to the total number of irreducible characters of $C_G(A)$.*

THEOREM 2.2. *Let G be solvable, let $P \in \text{Syl}_p(G)$ and let $N = N_G(P)$. Then G and N have equal numbers of irreducible characters with p' -degree.*

If A is solvable in Theorem 2.1, then (without assuming G is solvable), the conclusion follows by a result of G. Glauberman ([6] or see Chap. 13 of [9]). If A is nonsolvable, then by the odd order theorem, it follows that $|A|$ is even and thus $|G|$ is odd (and G is necessarily solvable). This case of Theorem 2.1 was proved in [1, 8]. We shall prove Theorem 2.1 here using Theorem B and without appealing to Glauberman's theorem. It should be mentioned, however, that both Glauberman's result when A is solvable and the author's proof in [8] when $|G|$ is odd yield explicit, uniquely defined bijections between the relevant sets of characters. (And T. R. Wolf has proved [12] that these maps agree when both are defined.) We shall not construct explicit maps in this paper.

Theorem 2.2 is the solvable case of a conjecture of J. McKay [10]. The case of Theorem 2.2 where $|G : N(P)|$ is odd was proved in [8] and the full (solvable) result was obtained by Wolf [13] using Dade's [3] result and also independently by Howlett. We will actually prove something slightly more general than (2.2) since we will allow P to be a Hall π -subgroup where π may consist of more than one prime.

We need some general (and known) facts about coprime action. The next result is due to Glauberman [5]. (Also see Lemma 13.8 and Corollary 13.9 of [9]). Its proof depends on the conjugacy part of the Schur-Zassenhaus theorem and so requires the Feit-Thompson theorem to prove it in full generality. We shall only need the case where G is solvable, however.

LEMMA 2.3. *Let groups A and G act on a set Ω and suppose also that A acts on G by automorphisms. Assume*

- (i) $(|A|, |G|)^{\perp} = 1$.
- (ii) G is transitive on Ω .
- (iii) $(a^g \cdot g) \cdot a = (a \cdot a) \cdot g^a$ for all $a \in \Omega$, $g \in G$ and $a \in A$.

Then the set of fixed points of A on Ω is nonempty and is an orbit under $C_G(A)$.

An example of how Glauberman's lemma can be applied in the study of characters is the following.

COROLLARY 2.4. *Let A act on G and let $N \triangleleft G$ be A -invariant. Assume that $(|A|, |G/N|) = 1$ and let $\chi \in \text{Irr}(G)$ be A -invariant. Let $B/N = C_{G/N}(A)$. Then χ_N has an A -invariant irreducible constituent θ and all such constituents are conjugate under B .*

Proof. Let Ω be the set of irreducible constituents of χ_N . Then G/N acts transitively on Ω and A acts on G/N and on Ω . The hypotheses of Lemma 2.3 hold and the result follows. ■

The next result is a somewhat less trivial application of Glauberman's lemma which is in some sense dual to Corollary 2.4. We shall only consider the case where G/N is abelian although the result holds without this assumption. (See Theorem 13.31 and Problem 13.10 of [9].)

LEMMA 2.5. *Let A act on G and let $N \triangleleft G$ be A -invariant. Assume that $(|A|, |G/N|) = 1$ and that G/N is abelian. Let $\theta \in \text{Irr}(N)$ be A -invariant. Then θ^G has an A -invariant irreducible constituent χ . If $C_{G/N}(A) = 1$, then χ is unique and if $C_{G/N}(A) = G/N$, then every irreducible constituent of θ^G is A -invariant.*

Proof. Let $H = \text{Irr}(G/N)$ and let Ω be the set of irreducible constituents of θ^G . Then A acts on H and on Ω and also H is a group and acts on Ω by $\chi \cdot \lambda = \chi\lambda$ for $\chi \in \Omega$ and $\lambda \in H$.

We show that H is transitive on Ω . If $\chi \in \Omega$, then $(\chi_N)^G = (\chi_N 1_N)^G = \chi(1_N)^G = \chi \sum_H \lambda$. If also $\psi \in \Omega$, then $0 \neq [\chi_N, \psi_N] = [(\chi_N)^G, \psi]$ and hence $\psi = \chi\lambda$ for some $\lambda \in H$ as desired.

Conditions (i) and (iii) of Lemma 2.3 are clear and we conclude that Ω contains an A -invariant element χ . To complete the proof we observe that if $C_{G/N}(A) = 1$, then $C_H(A) = 1$ and if $C_{G/N}(A) = G/N$, then $C_H(A) = H$. The result now follows from the fact that the A -fixed elements of Ω form an orbit under $C_H(A)$. ■

COROLLARY 2.6. *Let A act on G and let $N \triangleleft G$ be A -invariant. Assume that $(|A|, |G/N|) = 1$ and that $C_{G/N}(A) = G/N$. Let $\theta \in \text{Irr}(N)$ be A -invariant. Then every irreducible constituent of θ^G is A -invariant.*

Proof. Note that all conjugates of θ under G are A -invariant. Let $g \in G$ and write $M = \langle N, g \rangle$ so that M/N is abelian and M is A -invariant. If χ is a constituent of θ^G , then χ_M is a sum of irreducible characters, each of which is an irreducible constituent of some $(\theta^x)^M$ for $x \in G$. By Lemma 2.5, we see that each such character ψ is A -invariant and so $\psi(g^a) = \psi(g)$ for $a \in A$. Since this holds for all irreducible constituents of χ_M , we conclude that $\chi(g^a) = \chi(g)$. Since $g \in G$ was arbitrary, we see that χ is A -invariant as desired. ■

We need one further general lemma. (See Lemma 10.5 of [8].)

LEMMA 2.7. *Let $M \triangleleft G$ and $H \subseteq G$ with $MH = G$ and put $L = M \cap H$.*

Let $\theta \in \text{Irr}(M)$ be invariant in G and suppose that $\theta_L = \varphi \in \text{Irr}(L)$. Then restriction defines a bijection

$$\text{Irr}(G|\theta) \rightarrow \text{Irr}(H|\varphi).$$

Proof. Let $\chi \in \text{Irr}(G|\theta)$ and let ψ be an irreducible constituent of χ_H . Then $\chi_M = e\theta$, $\chi_L = e\varphi$ and $\psi_L = f\varphi$ for integers $f \leq e$. Now

$$e = [\chi_M, \theta] \leq [(\psi^G)_M, \theta] = [(\psi_L)^M, \theta] = [\psi_L, \varphi] = f \leq e$$

and so we have equality throughout.

From $e = f$, we conclude that $\chi(1) = \psi(1)$ and thus $\chi_H = \psi$ and restriction does define a map $\text{Irr}(G|\theta) \rightarrow \text{Irr}(H|\varphi)$. If also $\chi'_H = \psi$ with $\chi' \in \text{Irr}(G|\theta)$ and $\chi' \neq \chi$, then

$$[(\psi^G)_M, \theta] \geq [\chi_M, \theta] + [\chi'_M, \theta] > [\chi_M, \theta],$$

a contradiction. The map is thus one-to-one.

Given $\psi \in \text{Irr}(H|\varphi)$, we have

$$[(\psi^G)_M, \theta] = [(\psi_L)^M, \theta] = [\psi_L, \varphi] > 0$$

and so ψ^G has some constituent $\chi \in \text{Irr}(G|\theta)$. Then $\chi_H = \psi$ by the above and the map is onto. ■

We now work toward a proof of Theorem 2.1. If A acts on G , we introduce the notation $\text{Irr}_A(G)$ for the set of A -invariant irreducible characters of G . If $N \triangleleft G$ and $\theta \in \text{Irr}(N)$, then $\text{Irr}(G|\theta)$ is the set of irreducible constituents of θ^G . We write $\text{Irr}_A(G|\theta) = \text{Irr}_A(G) \cap \text{Irr}(G|\theta)$.

The next result is actually a slight generalization of Theorem 2.1. The original version follows from it by setting $L = 1$.

THEOREM 2.8. *Let A act on G with $(|G|, |A|) = 1$ and assume that G is solvable. Write $C = C_G(A)$ and let $L \triangleleft G$ be A -invariant. Let $\varphi \in \text{Irr}_A(L)$. Then*

$$|\text{Irr}_A(G|\varphi)| = |\text{Irr}_A(LC|\varphi)|.$$

Note that by Corollary 2.6, every element of $\text{Irr}(LC|\varphi)$ is A -invariant and so the subscript A on the right of the above equation is in fact redundant.

Proof of Theorem 2.8. We work by induction on $|G:L|$.

Step 1. We may assume that φ is invariant in G .

Proof. Let $T = I_G(\varphi)$, the inertia group, and suppose $T < G$. Note

that T is A -invariant since φ is, and so the inductive hypothesis yields $|\text{Irr}_A(T|\varphi)| = |\text{Irr}_A(L(T \cap C)|\varphi)|$.

Now character induction defines a bijection $\text{Irr}(T|\varphi) \rightarrow \text{Irr}(G|\varphi)$ (by Theorem 6.11 of [9]) and since this map commutes with the action of A , we have $|\text{Irr}_A(T|\varphi)| = |\text{Irr}_A(G|\varphi)|$ and similarly $|\text{Irr}_A(LC|\varphi)| = |\text{Irr}_A((LC \cap T)|\varphi)|$. Since $L(T \cap C) = LC \cap T$, the result follows in this case and so we may assume $T = G$. ■

Step 2. Let $L < K \triangleleft G$ with K A -invariant. Let C act on $\text{Irr}_A(K|\varphi)$ and let \mathcal{S} be a set of representatives for the orbits. Then for A -invariant X with $KC \subseteq X \subseteq G$, we have the disjoint union

$$\text{Irr}_A(X|\varphi) = \bigcup_{\theta \in \mathcal{S}} \text{Irr}_A(X|\theta). \quad (2.1)$$

Proof. Note that $\text{Irr}_A(K|\varphi)$ is invariant under C since C fixes φ by step 1. It is immediate that $\text{Irr}_A(X|\theta) \subseteq \text{Irr}_A(X|\varphi)$ for all $\theta \in \mathcal{S} \subseteq \text{Irr}(K|\varphi)$. Conversely, if $\xi \in \text{Irr}_A(X|\varphi)$, then by Corollary 2.4, ξ_K has some A -invariant irreducible constituent η . Since ξ_L is a multiple of φ by step 1, we have $\eta \in \text{Irr}_A(K|\varphi)$ and thus $\eta^c \in \mathcal{S}$ for some $c \in C$. Since $C \subseteq X$, it follows that $\xi \in \text{Irr}_A(X|\eta^c)$ and the equality in (2.1) is proved.

Now suppose $\xi \in \text{Irr}_A(X|\theta) \cap \text{Irr}_A(X|\theta')$ with $\theta, \theta' \in \mathcal{S}$. Then by Corollary 2.4, $\theta' = \theta^b$ for some $b \in B$, where $B/K = C_{X/K}(A)$. By a standard fact about coprime action (which follows easily from Lemma 2.3) we have $B = KC$ and $\theta' = \theta^c$ for some $c \in C$. Thus $\theta = \theta'$ by the definition of \mathcal{S} . ■

Step 3. We may assume that $KC = G$ for every A -invariant K with $L < K \triangleleft G$.

Proof. Assume that $KC < G$. By the inductive hypothesis we have $|\text{Irr}_A(LC|\varphi)| = |\text{Irr}_A(KC|\varphi)|$. Let \mathcal{S} be as in step 2. Then

$$|\text{Irr}_A(KC|\varphi)| = \sum_{\theta \in \mathcal{S}} |\text{Irr}_A(KC|\theta)|$$

by step 2 with $X = KC$.

Since $|G:K| < |G:L|$, the inductive hypothesis yields $|\text{Irr}_A(KC|\theta)| = |\text{Irr}_A(G|\theta)|$ for $\theta \in \mathcal{S}$. Another application of step 2, with $X = G$, yields

$$\sum_{\theta \in \mathcal{S}} |\text{Irr}_A(G|\theta)| = |\text{Irr}_A(G|\varphi)|.$$

Combining these equations yields the result. ■

Write $V = LC$. We may clearly assume $L \subseteq V < G$ and we fix A -invariant $K \triangleleft G$ such that K/L is a chief factor of the semidirect product GA .

Step 4. $KV = G$ and $K \cap V = L$.

Proof. The first assertion is immediate since $KC = G$ by step 3. Since K/L is abelian (by the solvability of G) it follows that $K \cap V \triangleleft G$ and thus $K \cap V \triangleleft GA$. If $K \cap V > L$, then since K/L is chief, we have $V \supseteq K \cap V = K$ and $G = KV = V$, a contradiction. ■

Step 5. We can write $\text{Irr}_A(K|\varphi) = \{\theta\}$.

Proof. We have $C_{K/L}(A) = L(K \cap C)/L = 1$ since $K \cap C \subseteq K \cap V = L$. Therefore, $|\text{Irr}_A(K|\varphi)| = 1$ by Lemma 2.5. ■

Step 6. Let θ be as in step 5. Then θ is invariant in GA and we may assume $\theta_L = e\varphi$ with $e^2 = |K:L|$.

Proof. Since C acts on $\text{Irr}_A(K|\varphi) = \{\theta\}$, we see that θ is invariant under $KC = G$ and thus is invariant in GA . By step 1 we have $\theta_L = e\varphi$ for some integer e and since K/L is chief in GA , the "going down" theorem ((6.18) of [9]) yields that either $e = 1$ or $e^2 = |K:L|$. If $e = 1$, then $|\text{Irr}(G|\theta)| = |\text{Irr}(V|\varphi)|$ by Lemma 2.7. However, $\text{Irr}(G|\theta) = \text{Irr}_A(G|\theta)$ and $\text{Irr}(V|\varphi) = \text{Irr}_A(V|\varphi)$ by Corollary 2.6. The result thus follows in this case and we may assume $e^2 = |K:L|$. ■

Step 7. We may assume that K/L is strong in G and K/L has a unique conjugacy class of complements in G/L .

Proof. If $K = G$ then $V = L$ and by step 5, $|\text{Irr}_A(G|\varphi)| = 1 = |\text{Irr}_A(V|\varphi)|$. Assume then that $K < G$ and let $N \triangleleft G$ be such that N/K is a chief factor of GA . If $U/L = C_{K/L}(N)$, then $U \triangleleft GA$ and so $U = K$ or $U = L$.

If $U = K$, then $N \cap V \triangleleft G$ and $N \cap V > L$. By step 3, $(N \cap V)C = G$ which is not the case since $(N \cap V)C \subseteq V < G$. Thus $U = L$ and $C_{K/L}(N) = 1$. Since N/K is a p -group, it follows that $(|K/L|, |N/K|) = 1$ and K/L is strong.

If H/L is a complement to K/L in G/L , then $(H \cap N)/L \in \text{Syl}_p(N/L)$. Also, $N_K(H \cap N) = U = L$ and so $H = N_G(H \cap N)$. It follows that all complements are conjugate. ■

Step 8. $|\text{Irr}_A(G|\varphi)| = |\text{Irr}_A(V|\varphi)|$, as desired.

Proof. By steps 6 and 7, Theorem B applies and there exists $H \subseteq G$ with $KH = G$ and $K \cap H = L$ such that $|\text{Irr}(G|\theta)| = |\text{Irr}(H|\varphi)|$. By step 7, we may assume $H = V$. Note that $\text{Irr}(G|\theta) = \text{Irr}_A(G|\theta)$ and $\text{Irr}(V|\varphi) = \text{Irr}_A(V|\varphi)$ by Corollary 2.6. Also $\text{Irr}_A(G|\theta) = \text{Irr}_A(G|\varphi)$ by steps 2 and 5. The result follows by combining these equations. ■

A proof of Theorem 2.2 can be carried out along quite similar lines. If π is a set of primes, we write $\text{Irr}^\pi(G)$ for the set of $\chi \in \text{Irr}(G)$ such that $\chi(1)$ is not divisible by any prime in π . Also, $\text{Irr}^\pi(G|\theta)$ denotes $\text{Irr}(G|\theta) \cap \text{Irr}^\pi(G)$ if $\theta \in \text{Irr}(N)$ with $N \triangleleft G$. The following is a generalization of Theorem 2.2.

THEOREM 2.9. *Let G be solvable and let $P \in \text{Syl}_\pi(G)$ be a Hall π -subgroup. Write $N = N_G(P)$. Let $L \triangleleft G$ and suppose $\varphi \in \text{Irr}_P(L)$. Then*

$$|\text{Irr}^\pi(G | \varphi)| = |\text{Irr}^\pi(LN | \varphi)|.$$

The next lemma substitutes for Corollary 2.4 in this situation.

LEMMA 2.10. *Let G be solvable and suppose $P \in \text{Syl}_\pi(G)$. Let $M \triangleleft G$ and $\chi \in \text{Irr}^\pi(G)$. Then χ_M has a constituent in $\text{Irr}_P(M)$ and all such constituents are conjugate under $N_G(P)$.*

Proof. Let θ be any irreducible constituent of χ_M and let $T = I_G(\theta)$. Since χ has π' -degree, we see that T has π' -index and thus $T^g \supseteq P$ for some $g \in G$. Replacing θ by θ^g and T by T^g , we may assume that $\theta \in \text{Irr}_P(M)$ as desired.

If also $\theta' \in \text{Irr}_P(M)$ is a constituent of χ_M , then $\theta' = \theta^x$ for some $x \in G$ and hence $P^x \subseteq I_G(\theta')$. Since also $P \subseteq I_G(\theta')$, we have $P^{xt} = P$ for some $t \in I_G(\theta')$. Then $xt \in N_G(P)$ and $\theta^{xt} = (\theta')^t = \theta'$. ■

Proof of Theorem 2.9. Most of this proof is essentially identical to the proof of Theorem 2.8 and so we shall provide only a sketch, emphasizing the differences from the previous argument.

The proof is by induction on $|G : L|$. We reduce to the case that φ is invariant in G using $T = I_G(\varphi)$. To prove that $|\text{Irr}^\pi(T | \varphi)| = |\text{Irr}^\pi(G | \varphi)|$, observe that T has π' -index since $P \subseteq T$.

Suppose $L < K \triangleleft G$. Then N acts on $\text{Irr}_P(K | \varphi)$ and we let \mathcal{S} be a set of orbit representatives. If $KN \subseteq X \subseteq G$, we have the disjoint union

$$\text{Irr}^\pi(X | \varphi) = \bigcup_{\theta \in \mathcal{S}} \text{Irr}^\pi(X | \theta). \quad (2.2)$$

The proof of this runs parallel to that of (2.1) in Theorem 2.8, except that Lemma 2.10 is used in place of Corollary 2.4.

Using the inductive hypothesis and two applications of (2.2) we obtain

$$|\text{Irr}^\pi(G | \varphi)| = |\text{Irr}^\pi(KN | \varphi)|$$

and thus if $KN < G$, the inductive hypothesis yields the result. We thus assume that $KN = G$ whenever $L < K \triangleleft G$.

Let $V = LN$ and assume $V < G$. Then $L < G$ and we may take K/L to be a chief factor of G . Then $KV = G$ and $K \cap V = L$. Also, by the Frattini argument, $V = N_G(PL)$ and thus $C_{K/L}(P) = 1$. Lemma 2.5 then yields that $\text{Irr}_P(K) = \{\theta\}$ for some θ . By (2.2) we have

$$\text{Irr}^\pi(G | \theta) = \text{Irr}^\pi(G | \varphi)$$

and it suffices to show that

$$|\text{Irr}^*(G|\theta)| = |\text{Irr}^*(V|\varphi)|. \quad (2.3)$$

Now θ is invariant in $KN = G$ and hence $\theta_L = e\varphi$ with $e = 1$ or $e^2 = |K:L|$. If $e = 1$, (2.3) follows from Lemma 2.7 since restriction preserves degrees. If $e^2 = |K:L|$ we observe that K/L is strong in G by taking $S = KP/C_{KP}(K/L)$. Since $C_{K/L}(P) = 1$, we have $C_{K/L}(S) = 1$. Also $(|S|, |K/L|) = 1$ as required.

Theorem B now yields $H \subseteq G$ with $HK = G$ and $H \cap K = L$. Replacing H by a conjugate, we may assume that $H \supseteq P$ and thus $H \supseteq V$. It follows that $H = V$ and Theorem B yields a bijection σ from $\text{Irr}(V|\varphi)$ onto $\text{Irr}(G|\theta)$. Since $\xi^\sigma(1) = e\xi(1)$, we see that σ maps $\text{Irr}^*(V|\varphi)$ onto $\text{Irr}^*(G|\theta)$ and (2.3) follows. ■

3. CHARACTER TRIPLE ISOMORPHISM

The purpose of this section is to present a result which will allow us to assume that $\varphi(1) = 1$ in the proof of Theorem B. The standard technique for this type of reduction is to use the theory of projective representations, which we find more convenient to apply indirectly. We will state a purely character theoretic result which is proved using projective representations. Appeal to this theorem eliminates the need for explicit use of projective representations in this paper and in many other applications. (The result in question appears as Theorem 11.28 of [9] and so we will not prove it here.)

We need some definitions. We say that (G, N, θ) is a *character triple* if $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ is invariant in G . With an appropriate definition of "isomorphism" (given below), we have

THEOREM 3.1. *Let (G, N, θ) be a character triple. Then there exists an isomorphic character triple (G^*, N^*, θ^*) such that $N^* \subseteq Z(G^*)$ and θ^* is faithful.*

Informally, what it means to say that (G, N, θ) and (G^*, N^*, θ^*) are isomorphic is that $G/N \cong G^*/N^*$ and that the characters lying over θ of the subgroups of G containing N behave "just like" the characters of the corresponding subgroups of G^* containing N^* . More precisely, suppose $\sigma: G/N \rightarrow G^*/N^*$ is an isomorphism. If $N \subseteq X \subseteq G$, let $X^* \subseteq G^*$ be defined by $\sigma(X/N) = X^*/N^*$. Suppose further that for each such X , there is a bijection (also denoted $*$) from $\text{Irr}(X|\theta)$ onto $\text{Irr}(X^*|\theta^*)$. To make the

various character bijections compatible with each other, with the isomorphism σ and with the characters of G/N and its subgroups, we assume

$$[\eta_H, \xi\beta] = [\eta_{H^*}^*, \xi^*\beta^\sigma] \quad (3.1)$$

whenever $N \subseteq H \subseteq K \subseteq G$; $\eta \in \text{Irr}(K | \theta)$; $\xi \in \text{Irr}(H | \theta)$ and $\beta \in \text{Irr}(H/N)$. Here, $\beta^\sigma \in \text{Irr}(H^*/N^*)$ is the character corresponding to β under σ (which maps H/N onto H^*/N^*). We say that (G, N, θ) and (G^*, N^*, θ^*) are *isomorphic* if this condition holds. (Actually, for the applications of Theorem 3.1 in this paper, a slightly weaker definition of character triple isomorphism would suffice. We will only need (3.1) in the case that β is the principal character of H/N .)

A useful (though trivial) example of a character triple isomorphism is $(G, N, \theta) \cong (G/K, N/K, \theta^*)$, where $K = \ker \theta$ and $\theta^*(Kn) = \theta(n)$. Some important consequences of an isomorphism between (G, N, θ) and (G^*, N^*, θ^*) are the following:

- (a) $\xi^*(1) = \xi(1) \theta^*(1)/\theta(1)$ for $\xi \in \text{Irr}(X | \theta)$.
- (b) If $N \subseteq X \subseteq Y \subseteq G$ and $\xi \in \text{Irr}(X | \theta)$, then ξ extends to Y iff ξ^* extends to Y^* .

Note that both the hypotheses and the conclusions of Theorem B are invariant under character triple isomorphisms $(G, L, \varphi) \cong (G^*, L^*, \varphi^*)$ and thus by Theorem 3.1 it is no loss to assume that $L \subseteq \mathbf{Z}(G)$ and φ is faithful (so that L is cyclic).

4. TENSOR INDUCTION

Let $H \subseteq G$ and let F be an arbitrary field. If we are given an $F[H]$ -module W , then the familiar process of induction yields the $F[G]$ -module W^G . In this section we give an exposition of "tensor induction" which is another, less well known, method for constructing an $F[G]$ -module from W . The resulting module is denoted $W^{\otimes G}$.

The construction of $W^{\otimes G}$ was given in 1967 by E. C. Dade (see pp. 180 and 181 of [1]) and was independently discovered by A. Dress [4] in 1970. A related, and in fact more general, construction was used at about the same time by J. P. Serre to prove a result on cohomological dimension of groups. (See p. 442 of [11] for an exposition of Serre's theorem.) Special cases of tensor induction have been used by T. R. Berger to study representations of particular types of groups.

To construct $W^{\otimes G}$, choose a set T of representatives for the right cosets of H in G . For $t \in T$ and $g \in G$, we write $t \cdot g \in T$ to denote the representative

of the coset Ht . Thus \cdot defines an action of G on T and $tg(t \cdot g)^{-1} \in H$ for all $t \in T$ and $g \in G$.

Now let $\{W_t \mid t \in T\}$ be a collection of F -spaces, each isomorphic to W , and let $\pi_t: W \rightarrow W_t$ be a fixed F -isomorphism. Write

$$W^{\otimes G} = \bigotimes_{t \in T} W_t$$

where we are assuming some fixed but arbitrary total order on T and the tensor product is over F . In order to make $W^{\otimes G}$ into a G -module, we define an action of $g \in G$ on the pure tensors $\bigotimes w_t$ for $w_t \in W_t$. The action will be well defined because it is linear in each w_t .

We write

$$(\bigotimes w_t)g = \bigotimes x_t,$$

where

$$x_t = w_{t \cdot g^{-1}} \pi^{-1}(t \cdot g^{-1}) g t^{-1} \pi \in W_t. \quad (4.1)$$

(Note that we have suppressed the subscripts $t \cdot g^{-1}$ on π^{-1} and t on π .) Since $w_{t \cdot g^{-1}} \pi^{-1} \in W$ and $(t \cdot g^{-1}) g t^{-1} \pi \in H$, we have $w_{t \cdot g^{-1}} \pi^{-1}(t \cdot g^{-1}) g t^{-1} \pi \in W$ and (4.1) makes sense. It is routine to check that $((\bigotimes w_t) g_1) g_2 = (\bigotimes w_t)(g_1 g_2)$ and thus $W^{\otimes G}$ is an $F[G]$ -module. Note that $\dim W^{\otimes G} = (\dim W)^{|G:H|}$. In particular, if $\dim W = 1$ then $\dim W^{\otimes G} = 1$. It is interesting to note that in this case, if W affords the linear character $\lambda: H \rightarrow F^\times$, then $W^{\otimes G}$ affords the character μ defined by $\mu(g) = \lambda(v(g))$ where $v: G \rightarrow H/H'$ is the transfer map.

Let $F = \mathbb{C}$ and suppose W affords the character ψ (where $\dim W$ is arbitrary). It is not too difficult to compute the character of $W^{\otimes G}$ and although this result will not be needed here, we will give a formula for this character which is denoted $\psi^{\otimes G}$. (See Proposition 9.20 of [1].) For $g \in G$, let T_0 be a set of representatives for the orbits of $\langle g \rangle$ in its action on T via \cdot . For $t \in T$, let $n(t)$ denote the size of the $\langle g \rangle$ -orbit containing t . Then

$$\psi^{\otimes G}(g) = \prod_{t \in T_0} \psi(tg^{n(t)}t^{-1}). \quad (4.2)$$

One can see from (4.2) that if T is replaced by another set of coset representatives for H in G or is given a different ordering, then the character $\psi^{\otimes G}$ is unchanged and thus the isomorphism class of the module $W^{\otimes G}$ depends only on the isomorphism class of W . (In fact, it is routine to prove this directly for arbitrary fields F .)

We will need the following character result.

LEMMA 4.1. *Let $N \subseteq H \subseteq G$ with $N \triangleleft G$ and let ψ be a character of H . Then for $n \in N$ we have*

$$\psi^{\otimes G}(n) = \prod_{t \in T} \psi(tnt^{-1}), \quad (4.3)$$

where T is a set of representatives for the right cosets of H in G .

Proof. Let W afford ψ and note that $t \cdot n = t$ for all $n \in N$. Thus in computing $(\otimes w_t)n = \otimes x_t$, (4.1) yields

$$x_t = w_t \pi_t^{-1} t n t^{-1} \pi_t$$

and so $(W^{\otimes G})_N$ is the tensor product of the representations W_t of N with characters ψ^t . ■

5. CENTRAL PRODUCTS

Let K be a group with subgroups W_1, \dots, W_r for $r > 1$ and let $L = \bigcap W_i$. We say that K is the (internal) *central product* of the W_i provided that

- (i) $[W_i, W_j] = 1$ for $i \neq j$.
- (ii) $\prod W_i = K$.
- (iii) $\prod (W_i/L)$ is direct in K/L .

Note that by (i) and (ii), each $W_i \triangleleft K$ and hence $L \triangleleft K$ and (iii) makes sense. Also, since $r > 1$, it follows from (i) and (ii) that $L \subseteq Z(K)$.

If K is the central product of subgroups W_i as above, we construct the external direct product $\hat{K} = \times W_i$. There exists a surjective homomorphism $\sigma: \hat{K} \rightarrow K$ defined by $\sigma((w_1, \dots, w_r)) = w_1 \cdots w_r$. The following result describing the character theory of central products is fairly standard.

LEMMA 5.1. *Assume the above notation and let $\theta \in \text{Irr}(K)$. Let θ_i be an irreducible constituent of θ_{w_i} and let $\hat{\theta} = \theta_1 \times \cdots \times \theta_r \in \text{Irr}(\hat{K})$. Then:*

- (a) *The θ_i are uniquely determined by θ .*
- (b) *$\hat{\theta}(x) = \theta(\sigma(x))$ for $x \in \hat{K}$.*
- (c) *$\ker \sigma \subseteq \ker \hat{\theta}$.*

Proof. Since $W_i C_K(W_i) = K$, each θ_i is invariant in K and we can write $\theta_{w_i} = a_i \theta_i$ for positive integers a_i . Statement (a) follows.

Because σ maps onto K , there exists $\psi \in \text{Irr}(\hat{K})$ with $\psi(x) = \theta(\sigma(x))$ for $x \in \hat{K}$. Since \hat{K} is a direct product, we can write $\psi = \psi_1 \times \cdots \times \psi_r$ with $\psi_i \in \text{Irr}(W_i)$. Conclusions (b) and (c) will follow when we prove $\psi_i = \theta_i$.

Let $b_i = \prod_{j \neq i} \psi_j(1)$. Then for $w \in W_i$ we have

$$b_i \psi_i(w) = \psi((1, \dots, w, 1, \dots, 1)) = \theta(w) = a_i \theta_i(w)$$

and since $\psi_i, \theta_i \in \text{Irr}(W_i)$, we can conclude by linear independence that $\psi_i = \theta_i$ as desired. ■

The following theorem, proved by the technique of tensor induction, is a generalization of a familiar fact about representations of wreath products: invariant irreducible representations of the base group are extendible.

THEOREM 5.2. *Let K be the central product of subgroups W_i , $1 \leq i \leq r$, and let U act on K and permute the W_i transitively. Let $G = KU$ be the semidirect product and let $\theta \in \text{Irr}(K)$ be invariant under U . Let θ_1 be the irreducible constituent of θ_w , and assume θ_1 is extendible to $W_1 H$, where $H = N_U(W_1)$. Then θ is extendible to G .*

Proof. Let \hat{K} and σ be as in the previous discussion. Define an action of U on $\{1, 2, \dots, r\}$ by $i \cdot u = j$ if $(W_i)^u = W_j$. Let U act on \hat{K} by

$$(w_1, \dots, w_r)^u = (w_{1 \cdot u}^u, \dots, w_{r \cdot u}^u).$$

It is routine to check that u defines an automorphism of \hat{K} and that $\sigma(x^u) = \sigma(x)^u$ for $x \in \hat{K}$. Let $\hat{G} = \hat{K}U$, the semidirect product, and let $\tau: \hat{G} \rightarrow G$ be defined by $\tau(xu) = \sigma(x)u$ for $x \in \hat{K}$ and $u \in U$. Then τ is a homomorphism which extends σ and maps \hat{G} onto G . By Lemma 5.1 suffices to extend $\hat{\theta}$ to \hat{G} .

Now let W be the natural image of W_1 in \hat{K} and let N be its complement, the product of the remaining W_i 's. Then $H = N_U(W)$ and we have $\hat{K}H = N_{\hat{G}}(W) = NWH$ and $N \cap (WH) = 1$. Note that the subgroup $WH \subseteq \hat{G}$ is isomorphic to $W_1 H \subseteq G$ and so viewing $\theta_1 \in \text{Irr}(W)$, we conclude that θ_1 is extendible to WH . It follows that there exists $\psi \in \text{Irr}(\hat{K}H)$ with $N \subseteq \ker \psi$ and $\psi_w = \theta_1$. Let $\chi = \psi^{\otimes \hat{\theta}}$. We claim that χ is the desired extension of $\hat{\theta}$ to \hat{G} .

Let T be a set of right coset representatives for H in U so that it also represents the cosets of $\hat{K}H$ in \hat{G} . For $x \in \hat{K}$, Lemma 4.1 yields

$$\chi(x) = \prod_{t \in T} \psi(x t t^{-1}).$$

Since $\psi((w_1, \dots, w_r)) = \theta_1(w_1)$ for $w_i \in W_i$, we see that

$$\chi((w_1, \dots, w_r)) = \prod_{t \in T} \theta_1(w_{1 \cdot t}^{t^{-1}}).$$

Because θ is invariant under U , we have $\theta_{1 \cdot u}(w^u) = \theta_1(w)$ for $w \in W_1$ and $u \in U$ and hence

$$\chi((w_1, \dots, w_r)) = \prod_{i \in T} \theta_{1 \cdot i}(w_{1 \cdot i}) = \prod_{i=1}^r \theta_i(w_i) = \hat{\theta}((w_1, \dots, w_r))$$

as desired. ■

The next result concerns a type of central semidirect product. It relates the existence of a character correspondence as in Theorem B with the existence of extensions of characters from normal subgroups.

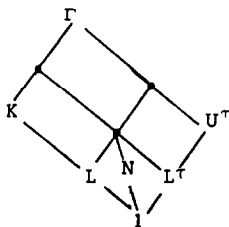
THEOREM 5.3. *Let $K \triangleleft G$ and $U \subseteq G$ with $KU = G$ and $K \cap U = L \subseteq Z(G)$. Observe that U/L acts on K and construct the semidirect product $K \rtimes (U/L) = G_0$. Let $\theta \in \text{Irr}(K)$ and write $\theta_L = e\varphi$ with $\varphi \in \text{Irr}(L)$.*

(a) *If θ is extendible to G_0 , then there exists a bijection $\pi: \text{Irr}(U| \varphi) \rightarrow \text{Irr}(G| \theta)$ with $\xi^\pi(1) = e\xi(1)$.*

(b) *If θ extends to G and φ extends to U , then θ extends to G_0 .*

Proof. Since U acts on K , we construct the semidirect product $K \rtimes U = \Gamma$. Identify K with the corresponding normal subgroup of Γ and let τ be a fixed isomorphism from U to a complement for K in Γ .

Note that $KL^\tau = K \times L^\tau$ and we can extend θ to $\hat{\theta} \in \text{Irr}(KL^\tau)$ with $L^\tau \subseteq \ker \hat{\theta}$. Now $\Gamma/L^\tau \cong G_0$ and θ extends to G_0 iff $\hat{\theta}$ extends to Γ .



We have a homomorphism $\sigma: \Gamma \rightarrow G$ defined by $\sigma(ku^\tau) = ku$. Then σ maps onto G and we let $N = \ker \sigma$ so that $N = \{l^{-1}l^\tau \mid l \in L\}$.

To prove (a), assume that θ extends to G_0 so that $\hat{\theta}$ extends to $\chi \in \text{Irr}(\Gamma)$. Let $\xi \in \text{Irr}(U| \varphi)$ and let $\tilde{\xi}$ be the corresponding character of Γ/K so that $\tilde{\xi}(ku^\tau) = \xi(u)$ for $k \in K$ and $u \in U$. By Theorem 6.17 of [9], $\xi \rightarrow \tilde{\xi}_\chi$ defines an injection of $\text{Irr}(U| \varphi)$ into $\text{Irr}(\Gamma| \theta)$. Note that $N \subseteq \ker(\tilde{\xi}_\chi)$ since $\tilde{\xi}(l^{-1}l^\tau) = \xi(l) = \xi(1)\varphi(l)$ and $\chi(l^{-1}l^\tau) = \hat{\theta}(l^{-1}l^\tau) = \theta(l^{-1}) = e\varphi(l^{-1}) = e\varphi(l)^{-1}$. Therefore $\tilde{\xi}_\chi$ may be viewed as a character ξ^π of $G \cong \Gamma/N$. For $k \in K$, we have $\xi^\pi(k) = \tilde{\xi}_\chi(k) = \xi(1)\theta(k)$ and hence $\xi^\pi \in \text{Irr}(G| \theta)$ and $\xi^\pi(1) = e\xi(1)$.

To see that π maps onto $\text{Irr}(G|\theta)$, let $\psi \in \text{Irr}(G|\theta)$ and view $\psi \in \text{Irr}(I|\theta)$ with $N \subseteq \ker \psi$. Then $\psi = \chi \xi$ for some $\xi \in \text{Irr}(U)$ and the condition that $N \subseteq \ker \psi$ yields that $\xi \in \text{Irr}(U|\varphi)$. Thus ψ lies in the image of π and the proof of (a) is complete.

To prove (b), assume that θ and φ extend to $\psi \in \text{Irr}(G)$ and $\xi \in \text{Irr}(U)$, respectively. View $\psi \in \text{Irr}(I)$ with $N \subseteq \ker \psi$ and let $\xi \in \text{Irr}(I/K)$ correspond to ξ as before. Note that $\xi(1) = 1$ and let $\chi = \psi \xi^{-1} \in \text{Irr}(I)$. Then

$$\chi(I^*) = \psi(I^*) \varphi(I)^{-1} = \psi(I) \varphi(I)^{-1} = e = \chi(1)$$

where the second equality holds since $I^{-1}I^* \in N \subseteq \ker \psi$. Thus $L^* \subseteq \ker \chi$ and χ is an extension of θ . Thus θ extends to G_0 as desired. ■

6. AN EXTENDIBLE CHARACTER

In this section we do most of the work toward proving Theorem B by showing that a certain character is extendible.

THEOREM 6.1. *Assume the following situation:*

- (i) $L \subseteq K \triangleleft G$ with $L \triangleleft G$ and K/L abelian.
- (ii) $U \subseteq G$ with $KU = G$ and $K \cap U = L$.
- (iii) $L \subseteq M \triangleleft U$ with M/L solvable, $(|K/L|, |M/L|) = 1$ and $c_{K/L}(M) = 1$.
- (iv) $\varphi \in \text{Irr}(L)$ is extendible to U .
- (v) $\theta \in \text{Irr}(K)$ with $\theta_L = e\varphi$, where $e^2 = |K/L|$.

Then θ is extendible to G .

We mention that the hypotheses easily imply that θ is invariant in G although this is not explicitly assumed. To see this, observe that φ is invariant in U and in K by (iv) and (v), respectively, and so φ is invariant in $G = KU$. Now $\varphi^K(1) = e^2\varphi(1) = e\theta(1)$ and $[\varphi^K, \theta] = [\varphi, \theta_L] = e$. Thus $\varphi^K = e\theta$ and this character is G -invariant since φ is. Therefore θ is invariant as claimed.

Before beginning the proof of Theorem 6.1, we need a few reasonably standard lemmas.

LEMMA 6.2. *Let $N \triangleleft G$ and let $\theta \in \text{Irr}(N)$ be invariant in G . If θ is not extendible to G , then there exists a Sylow subgroup P/N of G/N such that θ is not extendible to P . Furthermore, P/N is noncyclic.*

Proof. Replacing (G, N, θ) by an isomorphic character triple, we may

assume that $N \subseteq \mathbf{Z}(G)$. Since $\theta(1) = 1$, Theorem 6.26 of [9] applies and P exists with θ not extendible as claimed. In particular, P is not abelian and so P/N is not cyclic. ■

LEMMA 6.3. *Let $N \triangleleft G$ and let $\theta \in \text{Irr}(N)$ be invariant in G . Let $\chi \in \text{Irr}(G | \theta)$. Then $(\chi(1)/\theta(1))^2 \leq |G : N|$. In particular, $\chi(1)^2 \leq |G : \mathbf{Z}(G)|$ for all $\chi \in \text{Irr}(G)$.*

Proof. Let $\chi_N = e\theta$. Then $e\chi$ is a constituent of θ^G and comparison of degrees yields $e^2\theta(1) = e\chi(1) \leq \theta^G(1) = |G : N|\theta(1)$. ■

LEMMA 6.4. *Let $L = \mathbf{Z}(K)$ and assume that K/L is abelian. Let $\varphi \in \text{Irr}(L)$ be faithful and let $\theta \in \text{Irr}(K | \varphi)$. Then $\theta(1)^2 = |K : L|$ and $\text{Irr}(K | \varphi) = \{\theta\}$.*

Proof. If $\ker \theta > 1$, then since K is nilpotent we have $1 < L \cap \ker \theta \subseteq \ker \varphi = 1$, a contradiction. Thus θ is faithful and Theorem 2.31 of [9] yields that $\theta(1)^2 = |K : L|$. It follows that $\varphi^K = \theta(1)\theta$ and so θ is unique. ■

LEMMA 6.5. *Let $L \subseteq \mathbf{Z}(K)$ and assume that L is cyclic and K/L is abelian. Let $A \subseteq \text{Aut}(K)$ be such that $[K, A] \subseteq L$ and $[L, A] = 1$. Then $|A|$ divides $|K : L|$.*

Proof. Let $U = K/L$ and let $H = \text{Hom}(U, L)$. Make H into a group by defining multiplication pointwise. For $a \in A$, let $\varphi_a : U \rightarrow L$ be defined by $\varphi_a(\bar{x}) = [x, a]$, where $\bar{x} = Lx \in U$ for $x \in K$. Note that φ_a is a well defined element of H and that the map $a \rightarrow \varphi_a$ is an isomorphism from A into H . It therefore suffices to show that $|H|$ divides $|U|$.

Write $U \cong \prod C_i$, where the C_i are cyclic. Then $|H| = \prod |\text{Hom}(C_i, L)|$ and it suffices to show that $|\text{Hom}(C, L)|$ divides $|C|$ for cyclic groups C . In fact, since L is cyclic, it is easy to see that $|\text{Hom}(C, L)| = (|C|, |L|)$. ■

We are now ready to begin the proof of Theorem 6.1.

Proof of Theorem 6.1. We work by induction on $|G : L|$ and proceed in a number of steps.

Step 1. We may assume that φ is faithful and that $L = \mathbf{Z}(K) \subseteq \mathbf{Z}(G)$.

Proof. Since φ is invariant in G , we may apply Theorem 3.1 and find a character triple (G^*, L^*, φ^*) isomorphic to (G, L, φ) such that $L^* \subseteq \mathbf{Z}(G^*)$ and φ^* is faithful. For subgroups $X \subseteq G$ with $L \subseteq X$, let $X^* \subseteq G^*$ be the corresponding subgroup and for $\eta \in \text{Irr}(X, \varphi)$, let $\eta^* \in \text{Irr}(X^*, \varphi^*)$ be the corresponding character.

We claim that $G^*, K^*, L^*, U^*, M^*, \theta^*$ and φ^* satisfy conditions (i)–(v) in place of their corresponding objects. For conditions (i)–(iii), this is

immediate since $G^*/L^* \cong G/L$ and conditions (iv), (v) follow from the defining property (3.1) of character triple isomorphism. If we can prove that θ^* extends to G^* , then (3.1) yields that θ extends to G . It is therefore no loss to replace G by G^* since $|G:L| = |G^*:L^*|$ and the inductive nature of the proof is not affected.

Since $L^* \subseteq Z(G^*)$, what remains is to observe that $L^* = Z(K^*)$. This follows because

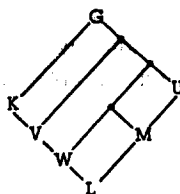
$$|K^*:L^*| \geq |K^*:Z(K^*)| \geq \theta^*(1)^2 = e^2 = |K^*:L^*|$$

where we have used Lemma 6.3. ■

Step 2. We may assume that K/L is a chief factor of G .

Proof. Let $L < W \subseteq K$ with W/L chief in G . First suppose W is abelian. Let $V = C_K(W)$ so that $W \subseteq V < G$. Now hypotheses (i)–(iii) of the theorem hold in the group VU with V , W , WU and WM in place of K , L , U and M . (That $C_{V/W}(WM) = 1$ follows, since $1 = C_{V/L}(M)$ covers $C_{V/W}(M)$ because $(|V/L|, |M/L|) = 1$.)

We proceed to find characters ϕ and θ of W and V so that conditions (iv) and (v) hold and we can apply the inductive hypothesis.



Now M/L acts on W and we have $(|M/L|, |W/L|) = 1$ and $C_{W/L}(M/L) = 1$. By Lemma 2.5, we conclude that ϕ^W has a unique M -invariant irreducible constituent $\hat{\phi}$ and since W is abelian, $\hat{\phi}$ extends ϕ . Since $M < U$, it follows from the uniqueness of $\hat{\phi}$, that $\hat{\phi}$ is invariant in WU . Now Lemma 2.7 applies and because ϕ extends to U , we conclude that $\hat{\phi}$ extends to WU . This is condition (iv) in VU .

Let θ be an irreducible constituent of $\hat{\phi}^V$. Then θ is a constituent of θ^K and hence $e = \theta(1) \leq \theta(1)|K:V|$. By Lemma 6.5 applied to W , we have $|K:V| \leq |W:L|$ and thus

$$|K:L| = e^2 \leq \theta(1)^2 |K:V| |W:L|.$$

Therefore, $\theta(1)^2 \geq |V:W|$ and since $W \subseteq Z(V)$, Lemma 6.3 yields $\theta(1)^2 = |V:W|$ and condition (v) holds in the group VU . Also, $\theta^K = \theta$. By the inductive hypothesis, θ extends to $\psi \in \text{Irr}(VU)$ and $(\psi^G)_K = (\psi^V)^K = \theta^K = \theta$. Thus ψ^G is the desired extension of θ .

Now assume W is nonabelian and that $W < K$. Since W/K is chief in G , we conclude that $L = Z(W)$. Let $\eta \in \text{Irr}(W|\varphi)$. By Lemma 6.4, we have $\eta_L = f\varphi$ with $f^2 = |W:L|$. We now see that conditions (i)–(v) hold in WU with W and η in place of K and θ and since $W < K$, the inductive hypothesis yields that η extends to WU .

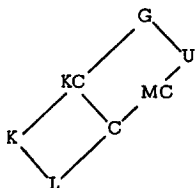
Since $\text{Irr}(W|\varphi) = \{\eta\}$ by Lemma 6.4, we have $\theta_w = a\eta$ for some integer a , and thus $|K:L| = \theta(1)^2 = a^2\eta(1)^2 = a^2|W:L|$ so that $|K:W| = a^2$. We now see that conditions (i)–(v) hold with W , WU , WM and η in place of L , U , M and φ and the inductive hypothesis yields that θ extends to G . We may thus assume $W = K$. ■

Step 3. We may assume that $C_U(K/L) = L$.

Proof. Let $C = C_U(K/L)$ so that $C \supseteq L$. By Lemma 6.5, it follows that $|C/C_C(K)|$ divides $|K/L|$. Since $C_C(K) \supseteq L$ and $(|M/L|, |K/L|) = 1$, we conclude that $M \cap C \subseteq C_C(K)$. Since $M, C \triangleleft U$, we have $[M, C] \subseteq M \cap C$ and hence $[M, C, K] = 1$. Also $[C, K] \subseteq L \subseteq Z(G)$ and so $[C, K, M] = 1$. The three subgroups lemma then yields $[K, M, C] = 1$.

Since K/L is abelian, it follows from condition (iii) that $[(K/L), M] = K/L$ and thus $L[K, M] = K$. It therefore follows from the preceding paragraph that C centralizes K .

If $C > L$, we wish to apply the inductive hypothesis with KC , C and MC in place of K , L and M . (Note that we do have $C \triangleleft G$ and conditions (i)–(iii) hold.) We need to find suitable characters $\hat{\varphi} \in \text{Irr}(C)$ and $\hat{\theta} \in \text{Irr}(KC)$.



Let ξ be an extension of φ to U and take $\hat{\varphi} = \xi_C$. Thus (iv) holds and we take $\hat{\theta}$ to be any irreducible constituent of $\hat{\varphi}^{KC}$. Now $\hat{\varphi}$ is invariant in KC since K centralizes C and so $\hat{\theta}_C = a\hat{\varphi}$ for some integer a with $a^2 \leq |KC:C| = e^2$ by Lemma 6.3. On the other hand, $\hat{\theta}_K$ is a constituent of $\varphi^K = e\theta$ and so $\hat{\theta}_K$ is a multiple of θ and $a = \hat{\theta}(1) \geq \theta(1) = e$. It follows that $a = e$ and $\hat{\theta}_K = \theta$.

If $C > L$, the inductive hypothesis yields an extension of $\hat{\theta}$ to G which is also an extension of θ . We may thus assume $C = L$. ■

Step 4. We may assume that M/L is elementary abelian.

Proof. Let M_0/L be a chief factor of U with $M_0 \subseteq M$. Since $M_0 \triangleleft U$, we see that $C_{K/L}(M_0)$ is invariant under U . By step 3, $C_{K/L}(M_0) <$

K/L and so by step 2 $C_{K/L}(M_0) = 1$. We may thus replace M by M_0 and assume that M/L is a chief factor of U . By condition (iii), M/L is solvable and so must be elementary abelian. ■

Step 5. We may assume that U/M is a noncyclic p -group where p is a prime dividing $|K/L|$.

Proof. Suppose θ does not extend to G . By Lemma 6.2 (since θ is invariant), there exists $S/K \in \text{Syl}_p(G/K)$ (for some prime p) such that θ does not extend to S and S/K is not cyclic. Thus every irreducible constituent of θ^S has degree divisible by p .

Let ζ be an extension of ϕ to U and observe that $((\zeta_{S \cap U})^S)_K = \phi^K = e\theta$ so that all irreducible constituents of $(\zeta_{S \cap U})^S$ lie in $\text{Irr}(S|\theta)$. It follows that $|K:L| = (\zeta_{S \cap U})^S(1)$ is divisible by p as desired.

By the inductive hypothesis, θ extends to every proper subgroup containing KM . Since θ does not extend to SM , we conclude that $SM = G$. Since $p \nmid |M/L|$, it follows that $U/M \cong G/KM \cong S/K$, a noncyclic p -group as claimed. ■

Step 6. M/L is noncyclic.

Proof. Let p be as in step 5 and let $P/L \in \text{Syl}_p(U/L)$. If M/L is cyclic, then it has prime order by step 4 and since P acts on M/L , this would imply that $P/C_P(M/L)$ is cyclic. Let $T = C_P(M/L)$. Since $U = MP$, it follows that $T \triangleleft U$ and thus $C_{K/L}(T)$ is U -invariant. Since T/L is a p -group and $p \mid |K/L|$, we have $C_{K/L}(T) > 1$ and thus $C_{K/L}(T) = K/L$ by step 2. But then $T = L$ by step 3 and P/L is cyclic. This contradicts step 5. ■

Step 7. There exist subgroups $W_i \subseteq K$ for $1 \leq i \leq r$ with $r > 1$ such that:

- (a) The W_i form an orbit under conjugation by U .
- (b) $\prod W_i = K$.
- (c) $M \subseteq N(W_i)$ for all i .
- (d) $[W_i, W_j] = 1$ for $i \neq j$.
- (e) $Z(W_i) = L$ for all i .

Proof. Let A/L be a simple M -submodule of K/L and let $N = C_M(A/L)$. Then $N < M$ and M/N is cyclic (since M/L is abelian). Thus $N > L$ by step 6 and we let $W/L = C_{K/L}(N) \cong A/L > 1$. Also, $W < K$ by step 3.

Let $W = W_1, W_2, \dots, W_r$ be the distinct U -conjugates of W . Then $r > 1$ since W cannot be invariant under U by step 2. Now $\prod W_i$ is U -invariant and so $\prod W_i = K$. Since $N \triangleleft M$, we have $M \subseteq N(W)$ and (c) follows. What remains is to prove (d) and (e).

Suppose $W^u \neq W$ for $u \in U$. Then $N^u \neq N$ and hence $NN^u = M$ since $|M : N|$ is prime. Thus $C_{W^u/L}(N) \subseteq C_{K/L}(M) = 1$ and so N acts with no fixed points on W^u/L . Since $(|N/L|, |W^u/L|) = 1$, it follows that $W^u = L[W^u, N]$.

Now $[N, W, W^u] \subseteq [L, W^u] = 1$ and since K/L is abelian, we have $[W, W^u, N] \subseteq [L, N] = 1$ so that $[W^u, N, W] = 1$ by the three subgroups lemma. Thus W centralizes $L[W^u, N] = W^u$ and (d) follows. Conclusion (e) is now immediate from (d) and (b) since $L = Z(K)$. ■

Step 8. θ does extend to G .

Proof. In the notation of step 7, K/L is the direct product of the W_i/L . To see this, suppose $\prod w_i = I \in L$ with $w_i \in W_i$. We must show that each $w_i \in L$. Now w_i centralizes W_j for $j \neq i$ and also w_i centralizes W_i since I and each w_j with $j \neq i$ do. Thus $w_i \in Z(K) = L$. It follows that K is the internal central product of the W_i .

Let $G_0 = K \rtimes (U/L)$. Suppose θ extends to G_0 . By Theorem 5.3(a) there is a bijection $\pi: \text{Irr}(U|\varphi) \rightarrow \text{Irr}(G|\theta)$ with $\xi^\pi(1) = e\xi(1)$. Taking ξ to be an extension of φ to U , ξ^π is the desired extension of θ to G .

What remains, is to show that θ does extend to G_0 . Let θ_1 be the (unique) irreducible constituent of θ_{W_1} and let $H = N_U(W_1)$. By Theorem 5.2, it suffices to show that θ_1 extends to $W_1 \rtimes (H/L)$. Since φ extends to H , it suffices by Theorem 5.3(b) to show that θ_1 extends to $W_1 H$.

To do this, we apply the inductive hypothesis to $W_1 H$ with W_1 , H and θ_1 in place of K , U and θ . (Note that we have $M \subseteq H$ by step 7(c).) Conditions (i)–(iv) are clear in this situation and condition (v) follows from Lemma 6.4 applied to W_1 . ■

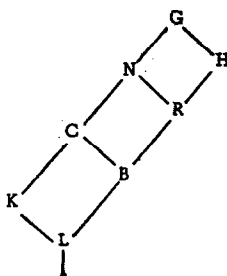
7. PROOF OF THEOREM B

Recall that the situation is the following. We have $L, K \triangleleft G$ with K/L a strong section of G . Also $\varphi \in \text{Irr}(L)$ and $\theta \in \text{Irr}(K)$ are invariant in G and $\theta_L = e\varphi$ with $e^2 = |K : L|$. We seek $H \subseteq G$ with $HK = G$ and $H \cap K = L$ and a bijection $\pi: \text{Irr}(H|\varphi) \rightarrow \text{Irr}(G|\theta)$.

We may replace the character triple (G, L, φ) with an isomorphic triple (G^*, L^*, φ^*) without changing either the hypotheses or the conclusion of the theorem. (Note that the fact that K/L is strong is determined in the group $G/L \cong G^*/L^*$.) We can therefore assume that $L \subseteq Z(G)$ and φ is faithful, so that L is cyclic. Also $L = Z(K)$ by Lemma 6.3 since $\theta(1)^2 = |K : L|$.

Now let $C = C_G(K/L)$ and $B = C_G(K)$. Then $B, C \triangleleft G$; $BK \subseteq C$ and $B \cap K = L$. By Lemma 6.5, $|C/B| \leq |K/L|$ and hence $BK = C$. Because K/L is a strong section, there exists $N \triangleleft G$ which induces a group S of automorphisms on K/L such that S is solvable, $(|K/L|, |S|) = 1$ and $C_{K/L}(S) = 1$. Replacing N by NC , it is no loss to assume that $N \supseteq C$ and we

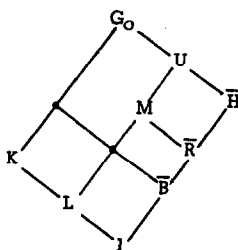
may identify N/C with S . Thus C/B is a normal Hall subgroup of N/B and so has a complement R/B . Let $H = N_G(R)$. Since all complements to C/B in N/B are conjugate, the Frattini argument yields that $NH = G$ and thus $KH = G$.



Now $[K \cap H, R] \subseteq K \cap R = L$ and thus $(K \cap H)/L \subseteq C_{K/L}(S) = 1$. Therefore $K \cap H = L$ as desired.

We must now show that there exists a bijection $\pi: \text{Irr}(H|\varphi) \rightarrow \text{Irr}(G|\theta)$ such that $\xi^\pi(1) = e\xi(1)$ for all $\xi \in \text{Irr}(H|\varphi)$. By Theorem 5.3(a) it suffices to show that θ is extendible to $G_0 = K \rtimes (H/L)$.

Let \bar{H} be the standard copy of H/L in G_0 and let \bar{R} and \bar{B} be the images of R and B in \bar{H} .



Since $B = C_G(K)$, we see that $K\bar{B} = K \times \bar{B}$ and thus θ and φ can be extended to $\bar{\theta} \in \text{Irr}(K\bar{B})$ and $\bar{\varphi} \in \text{Irr}(L\bar{B})$ with $\bar{B} \subseteq \ker \bar{\theta}$ and $\bar{B} \subseteq \ker \bar{\varphi}$. Then $\bar{\theta}_{L\bar{B}} = e\bar{\varphi}$ and $e^2 = |K : L| = |K\bar{B} : L\bar{B}|$. Also, $U = L\bar{H} = L \times \bar{H}$ and it follows that $\bar{\varphi}$ is extendible to U .

Next, let $M = L\bar{R}$. Then, $M/L\bar{B} \cong S$ and so is solvable and $(|M/L\bar{B}|, |K\bar{B}/L\bar{B}|) = 1$. Also $C_{K\bar{B}/L\bar{B}}(M) = 1$ since $C_{K/L}(S) = 1$. We see that the hypotheses of Theorem 6.1 are satisfied in G_0 with $K\bar{B}$, $L\bar{B}$, $\bar{\theta}$ and $\bar{\varphi}$ in place of K , L , θ and φ . Thus $\bar{\theta}$ (and hence θ) is extendible to G_0 . This completes the proof of Theorem B. ■

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